

A CONSISTENT DISCRETE VERSION OF A NON-AUTONOMOUS SIRVS MODEL

JOAQUIM P. MATEUS, CÉSAR M. SILVA, AND SANDRA VAZ

ABSTRACT. A discrete non-autonomous SIRVS model with general incidence is obtained from a continuous model by applying Mickens non-standard discretization method. Conditions for the permanence and extinction of the disease and the stability of disease-free solutions are obtained. It is also proved that, if the time step is sufficiently small, when we obtain extinction (respectively permanence) for the continuous model we also obtain extinction (respectively permanence) for the corresponding discrete model. An example showing the relevance of the consistency result is presented and numerical simulations are carried out to illustrate our findings.

1. INTRODUCTION

The law of mass action, that states that the rate of change in the disease incidence is directly proportional to the product of the number of susceptible and infective individuals, was the paradigm in the classic models in epidemiology. This is why classical models usually consider a bilinear incidence rate βSI , where S and I denote respectively the number of susceptible and infective individuals, to model the disease transmission. In spite of this, it is sometimes important to consider other forms for the incidence function.

Another usual assumption is that the model parameters are independent of time: in fact, the majority of the epidemiological models in the literature are given by a system of autonomous differential or difference equations. In spite of this, the assumption that the parameters are independent of time is not very realistic in many situations and it is useful to consider non-autonomous models that, for instance, allow

Date: July 25, 2016.

2010 Mathematics Subject Classification. 92D30, 37B55, 65L20.

Key words and phrases. Epidemic model, non-autonomous, global stability, numerical method.

Joaquim P. Mateus, César M. Silva and Sandra Vaz were partially supported by FCT through CMA-UBI (project UID/MAT/00212/2013).

the discussion of environmental and demographic effects that change with time [12, 13].

Most of the epidemiological models in the literature are continuous models [6], even though discrete-time models have also been addressed [1, 2, 3, 5, 11, 14, 15, 16, 18, 20, 25, 27, 31].

Naturally, there are several different ways of discretizing a continuous-time model. Recently, some models [7, 21, 23, 24, 28, 4] were discretized with the use of Mickens nonstandard finite difference schemes [19]. We note that all of these models are autonomous. Threshold conditions have been obtained for non-autonomous models in [37, 33].

In this paper we discuss a discrete non autonomous epidemic model with vaccination and incidence function given by some general function. Our model generalizes one that is obtained by Mickens nonstandard finite difference method from the continuous model [22] (see section 2). In [33], a discrete non-autonomous epidemic model with vaccination and mass action incidence was obtained by Mickens method. We note that, in the particular mass-action case, our model is not exactly similar to the model in [33], although Mickens rules were considered in both. We briefly compare computationally these two slightly different models in Section 6. The model we will consider is the following

$$(1) \quad \begin{cases} S_{n+1} - S_n = \Lambda_n - \beta_n \varphi(S_{n+1}, I_n) - (\mu_n + p_n)S_{n+1} + \eta_n V_{n+1} \\ I_{n+1} - I_n = \beta_n \varphi(S_n, I_n) + \sigma_n \psi(V_n, I_n) - (\mu_n + \alpha_n + \gamma_n)I_{n+1} \\ R_{n+1} - R_n = \gamma_n I_{n+1} - \mu_n R_{n+1} \\ V_{n+1} - V_n = p_n S_{n+1} - (\mu_n + \eta_n)V_{n+1} - \sigma_n \psi(V_{n+1}, I_n) \end{cases},$$

$n \in \mathbb{N}$, where the classes S , I , R , and V correspond, respectively, to susceptible, infective, recovered and vaccinated individuals and the parameter functions have the following meanings: Λ_n denotes the inflow of newborns in the Susceptible class; the function $\beta_n \varphi$ is the incidence (into the Infective class) from the susceptible individuals; the function $\sigma_n \psi$ is the incidence (into the Infective class) from the vaccinated individuals; μ_n are the natural deaths; p_n represents the vaccination of susceptibles; η_n represents the loss of immunity and consequence influx in the susceptible class; α_n are the deaths occurring in the infective class; γ_n is the recovery.

We will assume that $(\Lambda_n), (\mu_n), (p_n), (\eta_n), (\alpha_n), (\beta_n), (\sigma_n)$ and (γ_n) are bounded and nonnegative sequences and that there are positive constants $w_\mu, w_\Lambda, w_p, w_\gamma, M_\varphi$ and M_ψ such that:

- H1) the functions $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ are nonnegative and differentiable in $(\mathbb{R}_0^+)^2$ and the functions $\mathbb{R}_0^+ \ni x \rightarrow \partial_2 \varphi(x, 0)$ and $\mathbb{R}_0^+ \ni x \rightarrow \partial_2 \psi(x, 0)$ are non-decreasing and Lipschitz, with Lipschitz constants k_φ and k_ψ .
- H2) we have $\varphi(x, 0) = \psi(x, 0) = \varphi(0, y) = \psi(0, y) = 0$ for all $x, y \in \mathbb{R}_0^+$.

$$\text{H3)} \limsup_{n \rightarrow +\infty} \prod_{k=n}^{n+\omega_\mu} \frac{1}{1+\mu_k} < 1;$$

$$\text{H4)} \liminf_{n \rightarrow +\infty} \sum_{k=n+1}^{n+\omega_\Lambda} \Lambda_k > 0 \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \sum_{k=n+1}^{n+\omega_p} p_k > 0;$$

H5) Functions $\mathbb{R}^+ \ni y \mapsto \phi(x, y)/y$ and $\mathbb{R}^+ \ni y \mapsto \psi(x, y)/y$ are non-increasing.

It follows from H3) that there are constants $K > 0$ and $\theta \in]0, 1[$ such that

$$(2) \quad \prod_{k=m}^{n-1} \frac{1}{1+\mu_k} < K\theta^{n-m},$$

for $m, n \in \mathbb{N}$ sufficiently large. Additionally, using H1), H2) and H5), we have

$$\begin{aligned} \frac{\varphi(x, y)}{y} &= \frac{\varphi(x, y) - \varphi(x, 0)}{y - 0} \leq \frac{\varphi(x, y) - \varphi(x, 0)}{y - 0} \leq \partial_2 \varphi(x, 0) \\ &= |\partial_2 \varphi(x, 0) - \partial_2 \varphi(0, 0)| \leq k_\varphi x \end{aligned}$$

and thus

$$(3) \quad \varphi(x, y) \leq k_\varphi xy.$$

Similarly

$$(4) \quad \psi(x, y) \leq k_\psi xy$$

A natural question that arises when studying a discrete version of an existent continuous model is whether the dynamical behaviour of the models is identical for corresponding parameters, a problem sometimes referred to as “dynamical consistency”. In this work we prove that, when our conditions prescribe extinction (respectively permanence) for the continuous model we also have extinction (respectively permanence) for the corresponding discrete model as long as the time step is smaller than some constant (that depends on some parameters of the model and on the threshold condition). We also consider a family of examples of periodic system of period 1 such that the continuous and the discrete time system with time step $h = 1/L$ are not consistent, highlighting the importance of knowing that for time steps smaller than some explicit value we have consistency.

Finally, we carried out some simulations to illustrate our results.

2. DISCRETIZATION OF THE CONTINUOUS MODEL

We start with a non-autonomous SIRVS model that is slightly less general than the one considered in [22] and generalizes the one in [34]. Namely, we

consider the model:

$$(5) \quad \begin{cases} S' = \Lambda(t) - \beta(t)\varphi(S)I - (\mu(t) + p(t))S + \eta(t)V \\ I' = [\beta(t)\varphi(S) + \sigma(t)\psi(V) - \mu(t) - \alpha(t) - \gamma(t)]I \\ R' = \gamma(t)I - \mu(t)R \\ V' = p(t)S - (\mu(t) + \eta(t))V - \sigma(t)\psi(V)I \end{cases}.$$

We assume that the functions $\Lambda, \mu, p, \eta, \alpha, \beta, \sigma$ and γ belong to the class $C^1(\mathbb{R}_0^+)$, are nonnegative and bounded. We also require that:

C1) the functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ are nonnegative, non decreasing, differentiable and Lipschitz with Lipschitz constants k_φ and k_ψ respectively;

C2) $\varphi(0) = \psi(0) = 0$;

C3) there is $\omega > 0$ such that $\liminf_{t \rightarrow +\infty} \int_t^{t+\omega} \mu(s) ds > 0$.

In order to obtain threshold conditions for model (5), it was considered in [22] the following auxiliary system:

$$(6) \quad \begin{cases} x' = \Lambda(t) - [\mu(t) + p(t)]x + \eta(t)y \\ y' = p(t)x - [\mu(t) + \eta(t)]y. \end{cases}$$

and for each solution $(x^*(t), y^*(t))$ of (6) with positive initial conditions, it was shown that the numbers

$$R_C^\ell(\lambda) = \liminf_{t \rightarrow \infty} \int_t^{t+\lambda} \beta(s)\phi(x^*(s)) + \sigma(s)\psi(y^*(s)) - \mu(s) - \alpha(s) - \gamma(s) ds$$

and

$$R_C^u(\lambda) = \limsup_{t \rightarrow \infty} \int_t^{t+\lambda} \beta(s)\phi(x^*(s)) + \sigma(s)\psi(y^*(s)) - \mu(s) - \alpha(s) - \gamma(s) ds$$

are independent of the particular solution.

Using the above numbers, the following results are contained in results obtained in [22]:

Theorem 1 (Theorem 1 of [22]). *Assume that conditions C1), C2) and C3) hold. Then, if there is a constant $\lambda > 0$ such that $R_C^\ell(\lambda) > 0$ then the infectives I are permanent in system (5).*

Theorem 2 (Theorem 2 of [22]). *Assume that conditions C1), C2) and C3) hold. Then if there is a constant $\lambda > 0$ such that $R_C^u(\lambda) < 0$ then the infectives I go to extinction in system (5).*

Now, we will apply Micken's non-standard method to obtain a discrete version of system (5). Let $\phi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ be a positive continuous function such that

$$(7) \quad \lim_{h \rightarrow 0} \phi(h) = 0.$$

Given $h \in \mathbb{R}^+$, we let $t = nh$, with $n \in \mathbb{N}$, and identify $S'(t)$ with

$$\frac{S(nh + h) - S(nh)}{\phi(h)}.$$

After deciding a non-local representation for the incidence function and that terms that do not correspond to an interaction will be considered in the $n+1$ time, the first equation in (5) becomes

$$\begin{aligned} S((n+1)h) - S(nh) &= \phi(h) [\Lambda(nh) - \beta(nh)\varphi(S((n+1)h))I(nh) \\ &\quad - (\mu(nh) + p(nh))S((n+1)h) + \eta(nh)V(nh + h)]. \end{aligned}$$

Writing $S_n = S(nh)$, $I_n = I(nh)$, $V_n = V(nh)$, $\Lambda_n = \phi(h)\Lambda(nh)$, $\beta_n = \phi(h)\beta(nh)$, $\mu_n = \phi(h)\mu(nh)$, $p_n = \phi(h)p(nh)$ and $\eta_n = \phi(h)\eta(nh)$, we have

$$S_{n+1} - S_n = \Lambda_n - \beta_n\varphi(S_{n+1})I_n - (\mu_n + p_n)S_{n+1} + \eta_n V_{n+1}.$$

Proceeding similarly for the other equations, we obtain the following discrete model

$$(8) \quad \begin{cases} S_{n+1} - S_n = \Lambda_n - \beta_n\varphi(S_{n+1})I_n - (\mu_n + p_n)S_{n+1} + \eta_n V_{n+1} \\ I_{n+1} - I_n = \beta_n\varphi(S_n)I_n + \sigma_n\psi(V_n)I_n - (\mu_n + \alpha_n + \gamma_n)I_{n+1} \\ R_{n+1} - R_n = \gamma_n I_{n+1} - \mu_n R_{n+1} \\ V_{n+1} - V_n = p_n S_{n+1} - (\mu_n + \eta_n)V_{n+1} - \sigma_n\psi(V_{n+1})I_n \end{cases},$$

$n \in \mathbb{N}_0$. We will consider a model that contains this one to obtain some of our results. Namely, based on model (8), in sections 3 and 4 we will study model (1) that has a more general form for the incidence function.

Now, we need to make some definitions. We say that:

- i) the infectives (I_n) are *strong persistent* if for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 > 0$ there are constants $0 < m < M$ such that

$$m < \liminf_{t \rightarrow \infty} I_n \leq \limsup_{t \rightarrow \infty} I_n < M;$$

- ii) the infectives (I_n) go to *extinction* if for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 \geq 0$ we have $\lim_{t \rightarrow \infty} I_n = 0$.

Similar definitions can be made for the other compartments. For instance, if there exists constants $0 < m < M$ such that for any solution (S_n, I_n, R_n, V_n) of (1) with initial conditions $S_0, I_0, R_0, V_0 > 0$ we have

$$m < \liminf_{t \rightarrow \infty} S_n \leq \limsup_{t \rightarrow \infty} S_n < M$$

we say that the susceptibles are permanent.

3. AUXILIARY RESULTS

We need to consider the auxiliary system,

$$(9) \quad \begin{cases} x_{n+1} = \frac{\Lambda_n + \eta_n y_{n+1} + x_n}{1 + \mu_n + p_n} \\ y_{n+1} = \frac{p_n x_{n+1} + y_n}{1 + \mu_n + \eta_n} \end{cases}.$$

Note that the auxiliary system describes the behaviour of the system in the absence of infection. If (Λ_n) , (μ_n) , (p_n) , (η_n) , (α_n) , (μ_n) , (σ_n) and (β_n) are constant sequences then the linear system (9) becomes autonomous and corresponds to the linearization of the equations for (S_n) and (V_n) in the classical (autonomous) SIRVS model.

In order to proceed we need to recall some notions. A solution (u_n) of some system of difference equations $u_{n+1} = f_n(u_n)$ is said to be *attractive* if for all $n_0 \in \mathbb{N}$ and all $\varepsilon > 0$ there is $\sigma(n_0) > 0$ and $T(\varepsilon, n_0, u_0) \in \mathbb{N}$ such that if (\bar{u}_n) is a solution with $\|u_0 - \bar{u}_0\| < \sigma(n_0)$ then $\|u_n - \bar{u}_n\| < \varepsilon$, for all $n \geq n_0 + T(\varepsilon, n_0, u_0)$. Additionally, if some solution is attractive and we can take T to be independent of u_0 , we say that it is *uniformly attractive*.

The following theorem furnishes some simple properties of system (9).

Lemma 1 (Lemma 2.2 of [33]). *Assume that conditions H3) and H4) hold. Then*

- i) *all solutions (x_n, y_n) of system (9) with initial condition $x_0 \geq 0$ and $y_0 \geq 0$ are nonnegative for all $n \in \mathbb{N}_0$;*
- ii) *each fixed solution (x_n, y_n) of (9) is bounded and globally uniformly attractive for all $n \in \mathbb{N}_0$;*
- iii) *if (x_n, y_n) is a solution of (9) and $(\tilde{x}_n, \tilde{y}_n)$ is a solution of the system*

$$(10) \quad \begin{cases} x_{n+1} = \frac{\Lambda_n + \eta_n y_{n+1} + x_n + f_n}{1 + \mu_n + p_n} \\ y_{n+1} = \frac{p_n x_{n+1} + y_n + g_n}{1 + \mu_n + \eta_n} \end{cases}.$$

with $(\tilde{x}_0, \tilde{y}_0) = (x_0, y_0)$ then there is a constant $L > 0$, only depending on μ_n , satisfying

$$\sup_{n \in \mathbb{N}_0} \{|\tilde{x}_n - x_n| + |\tilde{y}_n - y_n|\} \leq L \sup_{n \in \mathbb{N}_0} (|f_n| + |g_n|);$$

- iv) *there exists constants $m, M > 0$ such that, for each solution (x_n, y_n) of (9), we have*

$$\begin{aligned} m &\leq \liminf_{t \rightarrow \infty} x_n \leq \limsup_{t \rightarrow \infty} x_n \leq M, \\ m &\leq \liminf_{t \rightarrow \infty} y_n \leq \limsup_{t \rightarrow \infty} y_n \leq M. \end{aligned}$$

- v) *when the system (9) is ω -periodic, it has a unique positive ω -periodic solution which is globally uniformly attractive.*

We have the following lemma

Lemma 2. Assume that condition H5) holds. Then we have the following:

- i) all solutions (S_n, I_n, R_n, V_n) of (1) with nonnegative initial conditions are nonnegative for all $n \in \mathbb{N}_0$;
- ii) all solutions (S_n, I_n, R_n, V_n) of (1) with positive initial conditions are positive for all $n \in \mathbb{N}_0$;
- iii) there is a constant $M > 0$ such that, if (S_n, I_n, R_n, V_n) is a solution of (1) with nonnegative initial conditions then

$$\limsup_{n \rightarrow +\infty} S_n < M \text{ and } \limsup_{n \rightarrow +\infty} V_n < M.$$

Proof. Let $S_n > 0$, $I_n > 0$, $R_n > 0$ and $V_n > 0$. By (1), (3) and (4), we obtain

$$V_{n+1} = \frac{V_n + p_n S_{n+1}}{1 + \mu_n + \eta_n + \sigma_n k_\psi I_n}$$

and thus

$$S_{n+1} \geq \frac{\eta_n(V_n + \Lambda_n + S_n) + (\Lambda_n + S_n)(1 + \mu_n + \sigma_n k_\psi I_n)}{(1 + \mu_n + \beta_n k_\varphi I_n)(1 + \mu_n + \eta_n + \sigma_n k_\psi I_n) + p_n(1 + \mu_n + \sigma_n k_\psi I_n)}.$$

Therefore we conclude that $S_{n+1} > 0$ and $V_{n+1} > 0$. By the second and third equations in (1) we obtain

$$I_{n+1} \geq \frac{I_n}{1 + \mu_n + \alpha_n + \gamma_n}$$

and

$$R_{n+1} = \frac{\gamma_n I_{n+1} + R_n}{1 + \mu_n}$$

and we conclude that $I_{n+1} > 0$ and $R_{n+1} > 0$. The previous inequalities allow us to conclude by induction that $S_n > 0$, $I_n > 0$, $R_n > 0$ and $V_n > 0$ for all $n \in \mathbb{N}$. In the same way we can conclude that, if $S_0 \geq 0$, $I_0 \geq 0$, $R_0 \geq 0$ and $V_0 \geq 0$, then $S_n \geq 0$, $I_n \geq 0$, $R_n \geq 0$ and $V_n \geq 0$ for all $n \in \mathbb{N}$. This proves i) and ii) in Lemma 2.

By (1), we have

$$\begin{cases} S_{n+1} \leq \frac{\Lambda_n + \eta_n V_{n+1} + S_n}{1 + \mu_n + p_n} \\ V_{n+1} \leq \frac{p_n S_{n+1} + V_n}{1 + \mu_n + \eta_n} \end{cases}.$$

By Lemma 1, we conclude that there is some $M > 0$ such that $S_n \leq x_n \leq M$ and $V_n \leq y_n \leq M$. The result is proved. \square

4. MAIN RESULTS

For each λ and each particular solution $\xi_n^* = (x_n^*, y_n^*)$ of (9) with $x_0^* > 0$ and $y_0^* > 0$ we define the numbers

$$(11) \quad \mathcal{R}_D^\ell(\xi^*, \lambda) = \liminf_{n \rightarrow \infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}$$

and

$$(12) \quad \mathcal{R}_D^u(\xi^*, \lambda) = \limsup_{n \rightarrow \infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k},$$

where $\partial_i f$ denotes the partial derivative of f with respect to the i -th variable. Contrarily to what one could expect, the next lemma shows that the numbers above do not depend on the particular solution $\xi_n = (x_n, y_n)$ of (9) with $x_n(0) > 0$ and $y_n(0) > 0$.

Lemma 3. *Assume that H1), H3) and H4) hold. If $(\xi_1^*)_n = ((x_1)_n^*, (y_1)_n^*)$ and $(\xi_2^*)_n = ((x_2)_n^*, (y_2)_n^*)$ are two solutions of (9) with $x_i^*(0) > 0$ and $y_i^*(0) > 0$, $i = 1, 2$, then*

$$\mathcal{R}_D^\ell(\xi_1^*, \lambda) = \mathcal{R}_D^\ell(\xi_2^*, \lambda), \quad \text{and} \quad \mathcal{R}_D^u(\xi_1^*, \lambda) = \mathcal{R}_D^u(\xi_2^*, \lambda).$$

Proof. To show that $\mathcal{R}_D^\ell(\xi^*, \lambda)$ is independent of the selection of $\xi^* = (x_n^*, y_n^*)$, a fixed solution of (9), it is important to note that according to ii) in Lemma 1, for any $\varepsilon > 0$ and any solution $\xi = (x_n, y_n)$ of system (9) with initial value $x_0 > 0$, $y_0 > 0$, there exists an $N \in \mathbb{N}^+$ such that, for $k \geq N$, we have $|x_k - x_k^*| \leq \varepsilon$ and $|y_k - y_k^*| \leq \varepsilon$. Hence

$$x_k^* - \varepsilon \leq x_k \leq x_k^* + \varepsilon \quad y_k^* - \varepsilon \leq y_k \leq y_k^* + \varepsilon.$$

By H1), we have

$$|\partial_2 \varphi(x_k, 0) - \partial_2 \varphi(x_k^*, 0)| \leq k_\varphi |x_k - x_k^*| \leq k_\varphi \varepsilon$$

and

$$|\partial_2 \psi(y_k, 0) - \partial_2 \psi(y_k^*, 0)| \leq k_\psi |y_k - y_k^*| \leq k_\psi \varepsilon.$$

So,

$$\partial_2 \varphi(x_k^*, 0) - k_\varphi \varepsilon \leq \partial_2 \varphi(x_k, 0) \leq \partial_2 \varphi(x_k^*, 0) + k_\varphi \varepsilon$$

and

$$\partial_2 \psi(y_k^*, 0) - k_\psi \varepsilon \leq \partial_2 \psi(y_k, 0) \leq \partial_2 \psi(y_k^*, 0) + k_\psi \varepsilon.$$

Combining the previous computations, we get

$$(13) \quad \begin{aligned} & \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0) - \bar{L}_k \varepsilon}{1 + \mu_k + \alpha_k + \gamma_k} \\ & \leq \frac{1 + \beta_k \partial_2 \varphi(x_k, 0) + \sigma_k \partial_2 \psi(y_k, 0)}{1 + \mu_k + \alpha_k + \gamma_k} \\ & \leq \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0) + \bar{L}_k \varepsilon}{1 + \mu_k + \alpha_k + \gamma_k}, \end{aligned}$$

where $\bar{L}_k = \beta_k k_\varphi + \sigma_k k_\psi$.

Let

$$r_k = \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}.$$

Using H1) and iv) in Lemma 1, it's easy to see that

$$r_k \leq \frac{1 + 2\beta^u k_\varphi M + 2\sigma^u k_\psi M}{1 + \mu^l + \alpha^l + \gamma^l} =: r,$$

for sufficiently large $k \in \mathbb{N}$, and that

$$\bar{L} = \frac{\bar{L}_k}{1 + \mu_k + \alpha_k + \gamma_k} \leq \frac{\beta^u k_\varphi + \sigma^u k_\psi}{1 + \mu^l + \alpha^l + \gamma^l} =: C.$$

So, for sufficiently large n ,

$$\prod_{k=n}^{n+\lambda} \left(r_k + \frac{\bar{L}_k \varepsilon}{1 + \mu_k + \alpha_k + \gamma_k} \right) \leq \prod_{k=n}^{n+\lambda} (r_k + C\varepsilon) = \prod_{k=n}^{n+\lambda} r_k + \Theta_\varepsilon$$

where

$$\Theta_\varepsilon = \binom{\lambda+1}{\lambda} r^\lambda C\varepsilon + \dots + \binom{\lambda+1}{1} r C^\lambda \varepsilon^\lambda + C^{\lambda+1} \varepsilon^{\lambda+1}.$$

Analogously

$$\prod_{k=n}^{n+\lambda} \left(r_k - \frac{\bar{L}_k \varepsilon}{1 + \mu_k + \alpha_k + \gamma_k} \right) \geq \prod_{k=n}^{n+\lambda} r_k - \Theta_\varepsilon.$$

By (13), we obtain

$$-\Theta_\varepsilon + \liminf_{n \rightarrow +\infty} \prod_{k=n}^{n+\lambda} r_k \leq \mathcal{R}_D^\ell(\xi^*, \lambda) \leq \Theta_\varepsilon + \liminf_{n \rightarrow +\infty} \prod_{k=n}^{n+\lambda} r_k.$$

Thus

$$\left| \mathcal{R}_D^\ell(\xi^*, \lambda) - \mathcal{R}_D^\ell(\xi, \lambda) \right| < \Theta_\varepsilon$$

and, by the arbitrariness of ε , we obtain $\mathcal{R}_D^\ell(\xi, \lambda) = \mathcal{R}_D^\ell(\xi^*, \lambda)$. Replacing \liminf by \limsup in the preceding argument, we reach a similar conclusion for $\mathcal{R}_D^u(\xi^*, \lambda)$. The result follows. \square

By lemma 3 we can drop the dependence of the particular solution and simply write $\mathcal{R}_D^\ell(\lambda)$ and $\mathcal{R}_D^u(\lambda)$ instead of $\mathcal{R}_D^\ell(\xi^*, \lambda)$ and $\mathcal{R}_D^u(\xi^*, \lambda)$ respectively.

We have the following result about the extinction of the disease:

Theorem 3 (Extinction of the disease). *Assume that conditions H1) to H5) hold. Then*

- a) *If there is a constant $\lambda > 0$ such that $\mathcal{R}_D^u(\lambda) < 1$, then the infectives (I_n) go to extinction.*
- b) *Any solution $(x_n^*, 0, 0, y_n^*)$, where (x_n^*, y_n^*) is a particular solution of system (9), is globally uniformly attractive.*

Proof. First note that the original system (1) can be rewritten as follows:

$$(14) \quad \begin{cases} S_{n+1} = \frac{1}{1+\mu_n+p_n} (\Lambda_n + S_n - \beta_n \varphi(S_{n+1}, I_n) + \eta_n V_{n+1}) \\ I_{n+1} = \frac{1}{1+\mu_n+\alpha_n+\gamma_n} (\beta_n \varphi(S_n, I_n) + \sigma_n \psi(V_n, I_n) + I_n) \\ R_{n+1} = \frac{1}{1+\mu_n} (\gamma_n I_{n+1} + R_n) \\ V_{n+1} = \frac{1}{1+\mu_n+\eta_n} (p_n S_{n+1} - \sigma_n \psi(V_{n+1}, I_n) + V_n) \end{cases},$$

$$n = 0, 1, \dots$$

Firstly, we will establish a). Since $\mathcal{R}_D^u(\lambda) < 1$, we can choose $\varepsilon_0 > 0$, $\varepsilon \in]0, 1[$ and a sufficiently large integer $N_1 \in \mathbb{N}$ such that

$$(15) \quad \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_k, 0) + \sigma_k \partial_2 \psi(y_k, 0) + (\beta^u k_\varphi + \sigma^u k_\psi) \varepsilon_0}{1 + \mu_k + \alpha_k + \gamma_k} < \varepsilon$$

for all $n \geq N_1$.

For any solution (S_n, I_n, R_n, V_n) of (14) with initial conditions $S_0 > 0$, $I_0 > 0$, $R_0 > 0$ and $V_0 > 0$, we have

$$(16) \quad \begin{cases} S_{n+1} \leq \frac{\Lambda_n + \eta_n V_{n+1} + S_n}{1 + \mu_n + p_n} \\ V_{n+1} \leq \frac{p_n S_{n+1} + V_n}{1 + \mu_n + \eta_n} \end{cases}.$$

By the comparison principle, we obtain $S_n \leq x_n$ and $V_n \leq y_n$ for all $n \in \mathbb{N}$, where (x_n, y_n) is the solution of (9) with initial condition $(x_0, y_0) = (S_0, V_0)$. According to Lemma 1, the solution (x_n^*, y_n^*) is globally uniformly attractive and thus, for the aforementioned $\varepsilon_0 > 0$, there exists an $N_2 \in \mathbb{N}$ such that

$$|x_n - x_n^*| \leq \varepsilon_0 \text{ and } |y_n - y_n^*| \leq \varepsilon_0 \text{ for all } n \geq N_2$$

From this, it may be concluded that

$$(17) \quad S_n \leq x_n^* + \varepsilon_0 \text{ and } V_n \leq y_n^* + \varepsilon_0 \text{ for all } n \geq N_2.$$

By the second equation of (1) we get

$$(18) \quad \begin{aligned} I_{n+1} &= \frac{1}{1 + \mu_n + \alpha_n + \gamma_n} (\beta_n \varphi(S_n, I_n) + \sigma_n \psi(V_n, I_n) + I_n) \\ &= \frac{1}{1 + \mu_n + \alpha_n + \gamma_n} \left(\beta_n \frac{\varphi(S_n, I_n)}{I_n} + \sigma_n \frac{\psi(V_n, I_n)}{I_n} + 1 \right) I_n. \end{aligned}$$

By H5), we have

$$(19) \quad \frac{\varphi(S_n, I_n)}{I_n} \leq \partial_2 \varphi(S_n, 0) \text{ and } \frac{\psi(V_n, I_n)}{I_n} \leq \partial_2 \psi(V_n, 0).$$

By H1), $x \mapsto \partial_2 \varphi(x, 0)$ and $x \mapsto \partial_2 \psi(x, 0)$ are non decreasing and also Lipschitz, so, using (17) we obtain

$$\begin{aligned} \partial_2 \varphi(S_n, 0) - \partial_2 \varphi(x_n^*, 0) &\leq \partial_2 \varphi(x_n^* + \varepsilon_0, 0) - \partial_2 \varphi(x_n^*, 0) \\ &= |\partial_2 \varphi(x_n^* + \varepsilon_0, 0) - \partial_2 \varphi(x_n^*, 0)| \\ &\leq k_\varphi \varepsilon_0 \end{aligned}$$

and thus

$$(20) \quad \partial_2 \varphi(S_n, 0) \leq \partial_2 \varphi(x_n^*, 0) + k_\varphi \varepsilon_0.$$

Analogously

$$(21) \quad \partial_2 \psi(V_n, 0) \leq \partial_2 \psi(y_n^*, 0) + k_\psi \varepsilon_0.$$

Therefore, by (15), (18), (19), (20) and (21) we have

$$I_{n+1} \leq \frac{\beta_n(\partial_2\varphi(x_n^*, 0) + k_\varphi\varepsilon_0) + \sigma_n(\partial_2\psi(y_n^*, 0) + k_\psi\varepsilon_0) + 1}{1 + \mu_n + \alpha_n + \gamma_n} I_n \leq \varepsilon I_n,$$

for all $n \geq N_2$. We conclude that $I_m \leq \varepsilon^{m-N_2} I_{N_2} \rightarrow 0$ as $m \rightarrow \infty$. This completes the proof of a).

Next, to establish b), let us consider two arbitrary solutions of the original system $(S_n^{(1)}, I_n^{(1)}, V_n^{(1)}, R_n^{(1)})$ and $(S_n^{(2)}, I_n^{(2)}, V_n^{(2)}, R_n^{(2)})$ and λ a constant such that $R_0^u(\lambda) < 1$. Let $\iota_n = I_n^{(1)} - I_n^{(2)}$ and $\rho_n = R_n^{(1)} - R_n^{(2)}$. By (8), we have

$$\begin{aligned} \rho_{n+1} - \rho_n &= (R_{n+1}^{(1)} - R_n^{(1)}) - (R_{n+1}^{(2)} - R_n^{(2)}) \\ &= \gamma_n(I_{n+1}^{(1)} - I_{n+1}^{(2)}) - \mu_n(R_{n+1}^{(1)} - R_{n+1}^{(2)}) \\ &= \gamma_n \iota_{n+1} - \mu_n \rho_{n+1}. \end{aligned}$$

Because $R_0^u(\lambda) < 1$, we conclude that $\iota_n \rightarrow 0$ as $n \rightarrow +\infty$ and therefore, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ sufficiently large such that, for $n \geq N$,

$$\rho_{n+1} + \mu_n \rho_{n+1} = \gamma_n \iota_{n+1} + \rho_n < \varepsilon + \rho_n$$

Thus, since $\mu_n > 0$, we get

$$\rho_{n+1} < \frac{\varepsilon}{1 + \mu_n} + \frac{\rho_n}{1 + \mu_n}.$$

and proceeding by induction

$$(22) \quad \rho_{n+1} < \left(\prod_{m=0}^{n-1} \frac{1}{1 + \mu_m} \right) \rho_0 + \varepsilon \sum_{m=0}^{n-1} \left(\prod_{k=m}^{n-1} \frac{1}{1 + \mu_k} \right).$$

By H3) and (2), we conclude that

$$\limsup_{n \rightarrow +\infty} \rho_n = 0.$$

Thus $\rho_n = R_n^{(1)} - R_n^{(2)} \rightarrow 0$ as $n \rightarrow +\infty$. Similar computations show that $S_n^{(1)} - S_n^{(2)} \rightarrow 0$ and $V_n^{(1)} - V_n^{(2)} \rightarrow 0$. This proves b) and the result follows. \square

We have the following result about the permanence of the disease:

Theorem 4 (Permanence of the disease). *Assume that conditions H1) to H5) hold. If there is a constant $\lambda > 0$ such that $\mathcal{R}_D^\ell(\lambda) > 1$ then the infectives (I_n) are strong persistent in system (1).*

Proof. Since $\mathcal{R}_0^\ell(\lambda) > 1$, there are $\varepsilon, \varepsilon_0 > 0$ such that

$$(23) \quad \liminf_{n \rightarrow +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0) - u}{1 + \mu_k + \alpha_k + \gamma_k} > 1 + \varepsilon,$$

for all $u \in [0, \varepsilon_0]$.

We claim that there is $\varepsilon_1 > 0$ such that

$$(24) \quad \limsup_{n \rightarrow \infty} I_n > \varepsilon_1$$

for every solution with positive initial conditions of system (1).

We proceed by contradiction. Assume that (24) doesn't hold. Then, for each ε_1 there is $N_1 \in \mathbb{N}$ and a solution (S_n, I_n, R_n, V_n) with positive initial conditions such that $I_n \leq \varepsilon_1$ for all $n \geq N_1$. By iii) in Lemma 2, we can assume that $S_n, V_n < M$ for all $n \geq N_1$. By (3) we have

$$(25) \quad \varphi(S_{n+1}, I_n) = k_\varphi S_{n+1} I_n \leq k_\varphi M \varepsilon_1$$

and likewise, by (4), we get

$$(26) \quad \psi(V_{n+1}, I_n) = k_\psi V_{n+1} I_n \leq k_\psi M \varepsilon_1.$$

By (1), (25) and (26) we have

$$(27) \quad \begin{cases} S_{n+1} \geq \frac{\Lambda_n + \eta_n V_{n+1} + S_n - \beta^u k_\varphi M \varepsilon_1}{1 + \mu_n + p_n} \\ V_{n+1} \geq \frac{p_n S_{n+1} + V_n - \sigma^u k_\psi M \varepsilon_1}{1 + \mu_n + \eta_n} \end{cases}.$$

for all $n \geq N_1$.

Given $\varepsilon_1 > 0$, consider the auxiliary system

$$(28) \quad \begin{cases} x_{n+1} = \frac{\Lambda_n + \eta_n y_{n+1} + x_n - \beta^u k_\varphi M \varepsilon_1}{1 + \mu_n + p_n} \\ y_{n+1} = \frac{p_n x_{n+1} + y_n - \sigma^u k_\psi M \varepsilon_1}{1 + \mu_n + \eta_n} \end{cases}.$$

For any $n_0 \in \mathbb{N}$ and $x_0, y_0 \in \mathbb{R}^+$, let (x_n, y_n) be the solution of (9) with initial condition $(x_{n_0}, y_{n_0}) = (x_0, y_0)$ and let $(\bar{x}_{n_0}, \bar{y}_{n_0}) = (x_0, y_0)$ be the solution of (28) with the same initial condition. By iii) in Lemma 1 we obtain

$$\sup_{n \in \mathbb{N}_0} \{|\bar{x}_n - x_n| + |\bar{y}_n - y_n|\} \leq LM(\beta^u k_\varphi + \sigma^u k_\psi) \varepsilon_1$$

and thus we can take $\varepsilon_1 > 0$ small enough such that

$$\sup_{n \in \mathbb{N}_0} \{|\bar{x}_n - x_n| + |\bar{y}_n - y_n|\} \leq \frac{\varepsilon_0}{2}.$$

On the other hand, by ii) in Lemma 1, there is $N_2 \geq N_1$ sufficiently large such that

$$|x_n^* - x_n| + |y_n^* - y_n| \leq \frac{\varepsilon_0}{2},$$

for all $n \geq N_2$. Therefore

$$(29) \quad |\bar{x}_n - x_n^*| + |\bar{y}_n - y_n^*| \leq \varepsilon_0,$$

for all $n \geq N_2$.

Noting that (27) can be written as

$$\begin{cases} S_{n+1} \geq \frac{\eta_n p_n (1 + \mu_n)(1 + \mu_n + p_n + \eta_n)(\Lambda_n + S_n - \beta^u M k_\varphi \varepsilon_1)}{(1 + \mu_n)(1 + \mu_n + p_n)(1 + \mu_n + p_n + \eta_n)} \\ \quad + \frac{(V_n - \sigma^u M k_\psi \varepsilon_1)(1 + \mu_n + p_n)}{(1 + \mu_n)(1 + \mu_n + p_n)(1 + \mu_n + p_n + \eta_n)} \\ V_{n+1} \geq \frac{p_n(\Lambda_n + S_n - \beta^u M k_\varphi \varepsilon_1) + (V_n - \sigma^u M k_\psi \varepsilon_1)(1 + \mu_n + p_n)}{(1 + \mu_n)(1 + \mu_n + p_n + \eta_n)} \end{cases}$$

and, using (28) and (29), we conclude that

$$S_n \geq \bar{x}_n \geq x_n^* - \varepsilon_0 \quad \text{and} \quad V_n \geq \bar{y}_n \geq y_n^* - \varepsilon_0,$$

for all $n \geq N_2$. Thus, since $S_n \leq x_n^*$ and $V_n \leq y_n^*$ for all $n \in \mathbb{N}$, we have

$$(30) \quad |S_n - x_n^*| \leq \varepsilon_0 \quad \text{and} \quad |V_n - y_n^*| \leq \varepsilon_0,$$

for all $n \geq N_2$. By H1), we have, for all $n \geq N_2$,

$$|\partial_2 \varphi(S_k, 0) - \partial_2 \varphi(x_k^*, 0)| \leq k_\varphi |S_k - x_k^*| \leq k_\varphi \varepsilon_0$$

and thus

$$(31) \quad \partial_2 \varphi(x_k^*, 0) - k_\varphi \varepsilon_0 \leq \partial_2 \varphi(S_k, 0) \leq \partial_2 \varphi(x_k^*, 0) + k_\varphi \varepsilon_0.$$

Reasoning similarly we obtain, for all $n \geq N_2$,

$$(32) \quad \partial_2 \psi(y_k^*, 0) - k_\psi \varepsilon_0 \leq \partial_2 \psi(V_k, 0) \leq \partial_2 \psi(y_k^*, 0) + k_\psi \varepsilon_0.$$

By H1) and H2), we conclude that

$$\varphi(S_n, I_n) = \varphi(S_n, 0) + \partial_2 \varphi(S_n, \varepsilon_n)(I_n - 0) = \partial_2 \varphi(S_n, \varepsilon_n) I_n,$$

for some $\varepsilon_n \in [0, \varepsilon_1]$, and all $n \geq N_2$. Thus, by continuity of $\partial_2 \varphi$,

$$\frac{\varphi(S_n, I_n)}{I_n} = \partial_2 \varphi(S_n, \varepsilon_n) \geq \partial_2 \varphi(S_n, 0) - \theta_1(\varepsilon_1),$$

with $\theta_1(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$, for all $n \geq N_2$. Thus, by (31)

$$\frac{\varphi(S_n, I_n)}{I_n} \geq \partial_2 \varphi(x_k^*, 0) - k_\varphi \varepsilon_0 - \theta_1(\varepsilon_1),$$

where $\theta_1(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$. Similarly, for all $n \geq N_2$, we have, by (32)

$$\frac{\psi(V_n, I_n)}{I_n} \geq \partial_2 \psi(y_k^*, 0) - k_\psi \varepsilon_0 - \theta_2(\varepsilon_1),$$

where $\theta_2(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$. From the second equation in (1), we have

$$(33) \quad \begin{aligned} I_{n+1} &= \frac{1 + \beta_n \varphi(S_n, I_n)/I_n + \sigma_n \psi(V_n, I_n)/I_n}{1 + \mu_n + \alpha_n + \gamma_n} I_n \\ &\geq \frac{1 + \beta_n \partial_2 \varphi(x_k^*, 0) + \sigma_n \partial_2 \psi(y_k^*, 0) - u}{1 + \mu_n + \alpha_n + \gamma_n} I_n. \end{aligned}$$

where

$$(34) \quad u = \beta^u(k_\varphi \varepsilon_0 + \theta_1(\varepsilon_1)) + \sigma^u(k_\psi \varepsilon_0 + \theta_2(\varepsilon_1)).$$

for all $n \geq N_2$. Letting ε_1 be sufficiently small, we conclude, according to (23) and (33), that $I_n \rightarrow +\infty$ as $n \rightarrow +\infty$. A contradiction. Thus, we conclude that (24) holds.

Next we will prove the strong persistence, namely we will see that there is a $\varepsilon_2 > 0$ such that, for any solution (S_n, I_n, R_n, V_n) of (1) with positive initial conditions, we have

$$(35) \quad \liminf_{n \rightarrow +\infty} I_n > \varepsilon_2.$$

Recall that, since $\mathcal{R}_0^\ell(\lambda) > 1$, there are $\varepsilon, \varepsilon_0 > 0$ such that (23) holds for all $u \in [0, \varepsilon_0]$.

If (35) doesn't hold, then, given $\varepsilon_0 > 0$, there must be a sequence of solutions of (1), $((S_{n,k}, I_{n,k}, R_{n,k}, V_{n,k})_{n \in \mathbb{N}})_{k \in \mathbb{N}}$, with initial conditions $(S_{0,k}, I_{0,k}, R_{0,k}, V_{0,k})$ such that

$$\liminf_{n \rightarrow +\infty} I_{n,k} < \frac{\varepsilon_0}{k^2}.$$

From (24), for each $k \in \mathbb{N}$, there must be two sequences $(s_{m,k})_{m \in \mathbb{N}}$ and $(t_{m,k})_{m \in \mathbb{N}}$ such that $s_{m,k} \rightarrow +\infty$ as $m \rightarrow +\infty$,

$$0 < s_{1,k} < t_{1,k} < s_{2,k} < t_{2,k} < \dots < s_{m,k} < t_{m,k} < \dots,$$

$$(36) \quad I_{s_{m,k},k} > \varepsilon_0/k, \quad I_{t_{m,k},k} < \varepsilon_0/k^2$$

and

$$\frac{\varepsilon_0}{k^2} \leq I_{n,k} \leq \frac{\varepsilon_0}{k}, \text{ for all } n \in [s_{m,k} + 1, t_{m,k} - 1] \cup \mathbb{N}.$$

Given $n \in [s_{m,k}, t_{m,k} - 1] \cup \mathbb{N}$, we have

$$\begin{aligned} I_{n+1,k} &= \frac{1 + \beta_n \varphi(S_{n,k}, I_{n,k})/I_{n,k} + \sigma_n \psi(V_{n,k}, I_{n,k})/I_{n,k}}{1 + \mu_n + \alpha_n + \gamma_n} I_{n,k} \\ &\geq \frac{1}{1 + \mu^u + \alpha^u + \gamma^u} I_{n,k} \end{aligned}$$

and therefore

$$(37) \quad \varepsilon_0/k^2 > I_{t_{n,k},k} \geq \sigma^{t_{n,k}-s_{n,k}} I_{s_{n,k},k} > \sigma^{t_{n,k}-s_{n,k}} \varepsilon_0/k,$$

where

$$\sigma = \frac{1}{1 + \mu^u + \alpha^u + \gamma^u}.$$

Thus, by (36) and (37),

$$(38) \quad t_{n,k} - s_{n,k} \geq \frac{\ln k}{\ln(1/\sigma)} \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

In view of (38), we can choose $k_0 \in \mathbb{N}$ such that

$$t_{n,k} - s_{n,k} > N + \lambda + 1,$$

for all $k \geq k_0$.

Letting m and $k \geq k_0$ be sufficiently large and $\varepsilon_2 > 0$ be sufficiently small, we may assume that

$$(39) \quad \begin{cases} S_{n,k} \geq \frac{\Lambda_n + \eta_n V_{n,k} + S_{n,k} - \beta^u k_\varphi M \varepsilon_2}{1 + \mu_n + p_n} \\ V_{n,k} \geq \frac{p_n S_{n,k} + V_{n,k} - \sigma^u k_\psi M \varepsilon_2}{1 + \mu_n + \eta_n} \end{cases}.$$

holds for all $n \in [s_{m,k} + 1, t_{m,k} - 1] \cap \mathbb{N}$.

Let (\bar{x}_n, \bar{y}_n) be a solution of (28) with initial value $(\bar{x}_{s_{m,k}}, \bar{y}_{s_{m,k}}) = (S_{s_{m,k}}, V_{s_{m,k}})$. We have $S_n \geq \bar{x}_n$ and $V_n \geq \bar{y}_n$ for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$. Letting $\varepsilon_2 > 0$ in (39) be sufficiently small and (x_n, y_n) be the solution of (9) with $x_{s_{m,k}+1} = S_{s_{m,k}+1}$ and $y_{s_{m,k}} = V_{s_{m,k}}$, we have by iii) in Lemma 1

$$|x_n - \bar{x}_n| + |y_n - \bar{y}_n| \leq \frac{\varepsilon_0}{2}$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$. we conclude that

$$S_n \geq \bar{x}_n \geq x_n - \varepsilon_0/2 > x_n^* - \varepsilon_0$$

and

$$V_n \geq \bar{y}_n \geq y_n - \varepsilon_0/2 > y_n^* - \varepsilon_0$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$.

Proceeding like before, we obtain (31) and (32) with S_k and V_k replaced by $S_{n,k}$ and $V_{n,k}$ respectively and, for sufficiently large $n \in \mathbb{N}$,

$$\frac{\varphi(S_{n,k}, I_{n,k})}{I_{n,k}} \geq \partial_2 \varphi(x_k^*, 0) - k_\varphi \varepsilon_0 - \theta_1(\varepsilon_1),$$

and

$$\frac{\psi(V_{n,k}, I_{n,k})}{I_{n,k}} \geq \partial_2 \psi(y_k^*, 0) - k_\psi \varepsilon_0 - \theta_2(\varepsilon_1),$$

where $\theta_1(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$. Therefore

$$I_{n+1,k} \geq \frac{1 + \beta_n \partial_2 \phi(x_n^*, 0) + \sigma_n \partial_2 \phi(y_n^*, 0) - u}{1 + \mu_n + \alpha_n + \gamma_n} I_{n,k},$$

for all $n \in [s_{m,k}, t_{m,k}] \cap \mathbb{N}$, where u is given by (34). Thus

$$\frac{\varepsilon_0}{k^2} > I_{t_{m,k},k} \geq I_{t_{m,k}-\lambda,k} \prod_{n=t_{m,k}-\lambda}^{t_{m,k}} \frac{1 + \beta_n \partial_2 \phi(x_n^*, 0) + \sigma_n \partial_2 \phi(y_n^*, 0) - u}{1 + \mu_n + \alpha_n + \gamma_n} > \frac{\varepsilon_0}{k^2}$$

which is a contradiction. The theorem follows. \square

We consider now the particular periodic case: assume that all parameters of system (1) are periodic with period $\omega \in \mathbb{N}$. By v) in Lemma 1, there is an ω -periodic disease-free solution of (9), $\xi^* = (x_n^*, y_n^*)_{n \in \mathbb{N}}$. Thus, in the periodic setting, (11) and (12) become both equal to

$$(40) \quad \mathcal{R}_D^{per}(\xi^*) = \prod_{k=0}^{\omega-1} \frac{1 + \beta_k \partial_2 \varphi(x_k^*, 0) + \sigma_k \partial_2 \psi(y_k^*, 0)}{1 + \mu_k + \alpha_k + \gamma_k}.$$

Therefore we obtain the corollary:

Corollary 1 (Periodic case). *Assume that all coefficients are ω -periodic in (1) and that conditions H1) to H5) hold. Then*

- a) *If $\mathcal{R}_D^{per}(\xi^*) < 1$ then the infectives (I_n) go to extinction.*
- b) *The disease-free solution $(x_n^*, 0, 0, y_n^*)$, where $(x_n^*, y_n^*)_{n \in \mathbb{N}}$ is an disease-free ω -periodic solution of (9), is globally attractive.*
- c) *If $\mathcal{R}_D^{per}(\xi^*) > 1$ then the infectives (I_n) are permanent.*

5. CONSISTENCY

In this section, under the additional assumption that the parameter functions Λ , μ , η and p are constant, we will obtain a result that states that when our integral conditions prescribe extinction (respectively persistence) for the continuous-time model, then the discrete-time conditions prescribe extinction (respectively persistence) for the corresponding discrete-time models, as long as the time step is less than some constant.

We assume in this section that the parameter functions Λ , μ , η and p are constant functions and let $\phi(h)$ be the function used in the discretization of the derivative.

We consider the continuous time model (5) and, for a given time step h , the corresponding discrete time model, that is the discrete time model with parameters $\beta_k^h = \phi(h)\beta(kh)$, $\sigma_k^h = \phi(h)\sigma(kh)$, $\Lambda_k^h = \phi(h)\Lambda$, $\mu_k^h = \phi(h)\mu$, $p_k^h = \phi(h)p$, $\eta_k^h = \phi(h)\eta$, $\alpha_k^h = \phi(h)\alpha(kh)$ and $\gamma_k^h = \phi(h)\gamma(kh)$.

For a given time step $h > 0$, the expressions $\mathcal{R}_D^\ell(\lambda)$ and $\mathcal{R}_D^u(\lambda)$ in (11) and (12) become, in our context

$$\mathcal{R}_D^\ell(\lambda, h) = \liminf_{n \rightarrow \infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k^h \varphi(x_{k+1}^*) + \sigma_k^h \psi(y_{k+1}^*)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h}$$

and

$$\mathcal{R}_D^u(\lambda, h) = \limsup_{n \rightarrow \infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k^h \varphi(x_{k+1}^*) + \sigma_k^h \psi(y_{k+1}^*)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h},$$

where (x_k^*, y_k^*) is the solution of the (in our context autonomous) system (9).

We have the following result.

Theorem 5. *For system (5), assume that $\Lambda(t) = \Lambda$, $\mu(t) = \mu$, $\eta(t) = \eta$ and $p(t) = p$ for all $t \geq 0$ and that the functions $\alpha(t)$, $\gamma(t)$, $\beta(t)$ and $\sigma(t)$ are differentiable, nonnegative, bounded and have bounded derivative. Assume also that conditions C1) to C3) hold and let*

$$h_{max}^u = -\frac{R_C^u}{\sup_{t \geq 0} |f'(t)|(\lambda + 1)} \quad \text{and} \quad h_{max}^\ell = \frac{R_C^\ell}{\sup_{t \geq 0} |f'(t)|(\lambda + 1)},$$

where

$$f(t) = \beta(t)\varphi\left(\frac{\Lambda(\mu + \eta)}{\mu(\mu + \eta + p)}\right) + \sigma(t)\psi\left(\frac{p\Lambda}{\mu(\mu + \eta + p)}\right) - \mu - \alpha(t) - \gamma(t).$$

Then:

- a) If $\mathcal{R}_C^u(\lambda) < 0$ then $\mathcal{R}_D^u(\lfloor \lambda/h \rfloor, h) < 1$ for all $h \in]0, h_{max}^u[$;
b) If $\mathcal{R}_C^\ell(\lambda) > 0$ then $\mathcal{R}_D^\ell(\lfloor \lambda/h \rfloor, h) > 1$ for all $h \in]0, h_{max}^\ell[$.

Proof. Observe that $(x_n, y_n) = (a, b)$, $n \in \mathbb{N}$ and $(x(t), y(t)) = (a, b)$, $t \in \mathbb{R}$, where

$$(a, b) = (\Lambda(\mu + \eta)/[\mu(\mu + \eta + p)], p\Lambda/[\mu(\mu + \eta + p)]),$$

are respectively solutions of system (9) and system (6). Thus

$$\mathcal{R}_C^u(\lambda) = \limsup_{t \rightarrow \infty} \int_t^{t+\lambda} \beta(s)\varphi(a) + \sigma(s)\psi(b) - [\mu(s) + \alpha(s) + \gamma(s)] ds$$

and

$$\mathcal{R}_D^u(\lfloor \lambda/h \rfloor, h) = \limsup_{n \rightarrow \infty} \prod_{k=n}^{n+\lfloor \lambda/h \rfloor} \frac{1 + \beta_k^h \varphi(a) + \sigma_k^h \psi(b)}{1 + \mu_k^h + \alpha_k^h + \gamma_k^h}.$$

By contradiction, assume that

$$(41) \quad \mathcal{R}_C^u(\lambda) < 0,$$

and that there is a sequence $(h_m)_{m \in \mathbb{N}}$ such that $h_m \rightarrow 0$ as $m \rightarrow +\infty$ and

$$(42) \quad \mathcal{R}_D^u(\lfloor \lambda/h_m \rfloor, h_m) = \limsup_{n \rightarrow \infty} \prod_{k=n}^{n+\lfloor \lambda/h_m \rfloor} \frac{1 + \beta_k^{h_m} \varphi(a) + \sigma_k^{h_m} \psi(b)}{1 + \mu_k^{h_m} + \alpha_k^{h_m} + \gamma_k^{h_m}} \geq 1,$$

for all $m \in \mathbb{N}$. By (42), we conclude that, for each $m \in \mathbb{N}$, there are sequences $(h_m)_{m \in \mathbb{N}}$ and $(n_{m,r})_{r \in \mathbb{N}}$ such that $h_m \rightarrow 0$ as $m \rightarrow +\infty$, $n_{m,r} \rightarrow +\infty$ as $r \rightarrow +\infty$ and

$$(43) \quad \begin{aligned} & \prod_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m \rfloor} (1 + \beta_k^{h_m} \varphi(a) + \sigma_k^{h_m} \psi(b)) \\ & > (1 - h_m) \prod_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m \rfloor} (1 + \mu_k^{h_m} + \alpha_k^{h_m} + \gamma_k^{h_m}). \end{aligned}$$

By (43), we have

$$(44) \quad \begin{aligned} & \sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m \rfloor} (\beta_k^{h_m} \varphi(a) + \sigma_k^{h_m} \psi(b) - \mu_k^{h_m} - \alpha_k^{h_m} - \gamma_k^{h_m}) \\ & > B_{n_{m,r}, \lambda, h_m} - A_{n_{m,r}, \lambda, h_m} - C_{n_{m,r}, \lambda, h_m} \end{aligned}$$

and, multiplying both sides by h_m , we get

$$(45) \quad \begin{aligned} & \phi(h_m) \sum_{k=n_{m,r}}^{n_{m,r}+\lfloor \lambda/h_m \rfloor} h_m [\beta(kh_m) \varphi(a) + \sigma(kh_m) \psi(b) - \mu(kh_m) - \alpha(kh_m) - \gamma(kh_m)] \\ & > B_{n_{m,r}, \lambda, h_m} - A_{n_{m,r}, \lambda, h_m} - C_{n_{m,r}, \lambda, h_m}, \end{aligned}$$

where

$$(46) \quad A_{n,L,h} := -h + h \prod_{k=n}^{n+\lfloor L/h \rfloor} (1 + \beta_k^h \varphi(a) + \sigma_k^h \psi(b)) - h \sum_{k=n}^{n+\lfloor L/h \rfloor} (\beta_k^h \varphi(a) + \sigma_k^h \psi(b)),$$

$$(47) \quad B_{n,L,h} := -h + h \prod_{k=n}^{n+\lfloor L/h \rfloor} (1 + \mu_k^h + \alpha_k^h + \gamma_k^h) - h \sum_{k=n}^{n+\lfloor L/h \rfloor} (\mu_k^h + \alpha_k^h + \gamma_k^h)$$

and

$$(48) \quad C_{n,L,h} = h \prod_{k=n}^{n+\lfloor L/h \rfloor} (1 + \mu_k^h + \alpha_k^h + \gamma_k^h).$$

We have

$$(49) \quad \begin{aligned} |A_{n_m,r,\lambda,h_m}| &\leq h_m \sum_{k=2}^{\lfloor \lambda/h_m \rfloor} \binom{\lfloor \lambda/h_m \rfloor}{k} [(\beta^u \varphi(a) + \sigma^u \psi(b))]^k [\phi(h_m)]^k \\ &\leq h_m \sum_{k=0}^{\lfloor \lambda/h_m \rfloor} \binom{\lfloor \lambda/h_m \rfloor}{k} [(\beta^u \varphi(a) + \sigma^u \psi(b))]^k [\phi(h_m)]^k \\ &= h_m [1 + (\beta^u \varphi(a) + \sigma^u \psi(b))\phi(h_m)]^{\lfloor \lambda/h_m \rfloor}. \end{aligned}$$

Noting that, by (7), we have

$$\begin{aligned} &\lim_{m \rightarrow +\infty} [1 + (\beta^u \varphi(a) + \sigma^u \psi(b))\phi(h_m)]^{\lfloor \lambda/h_m \rfloor} \\ &= \lim_{m \rightarrow +\infty} \left[\left(1 + \frac{\beta^u \varphi(a) + \sigma^u \psi(b)}{1/\phi(h_m)} \right)^{1/\phi(h_m)} \right]^{\phi(h_m) \lfloor \lambda/h_m \rfloor} \\ &= e^{(\beta^u \varphi(a) + \sigma^u \psi(b))\lambda} \end{aligned}$$

and that a convergent sequence is bounded, by (49) there is $C_1 > 0$ such that

$$(50) \quad |A_{n_m,r,\lambda,h_m}| \leq C_1 h_m.$$

Similarly, we have

$$(51) \quad \begin{aligned} |B_{n_m,r,\lambda,h_m}| &\leq \sum_{k=2}^{\lfloor \lambda/h_m \rfloor} \binom{\lfloor \lambda/h_m \rfloor}{k} [(\mu^u + \alpha^u + \gamma^u)]^k [\phi(h_m)]^k \\ &\leq \sum_{k=0}^{\lfloor \lambda/h_m \rfloor} \binom{\lfloor \lambda/h_m \rfloor}{k} [(\mu^u + \alpha^u + \gamma^u)]^k [\phi(h_m)]^k \\ &= h_m [1 + (\mu^u + \alpha^u + \gamma^u)\phi(h_m)]^{\lfloor \lambda/h_m \rfloor}. \end{aligned}$$

Using (7) again, we get

$$\lim_{m \rightarrow +\infty} [1 + (\mu^u + \alpha^u + \gamma^u)\phi(h_m)]^{\lfloor \lambda/h_m \rfloor} \leq e^{(\mu^u + \alpha^u + \gamma^u)\lambda},$$

there is $C_2 > 0$ such that

$$(52) \quad |B_{n_m,r,\lambda,h_m}| \leq C_2 h_m.$$

Finally, we have

$$\begin{aligned} |C_{n_m, r, \lambda, h_m}| &= h_m \prod_{k=n_m, r}^{n_m, r + \lfloor \lambda/h_m \rfloor} (1 + \mu_k^{h_m} + \alpha_k^{h_m} + \gamma_k^{h_m}) \\ &= h_m (1 + 3\phi(h_m) \max\{\mu^u, \alpha^u, \gamma^u\})^{\lfloor \lambda/h_m \rfloor + 1}. \end{aligned}$$

According (7), we obtain

$$\begin{aligned} &\lim_{m \rightarrow +\infty} (1 + 3\phi(h_m) \max\{\mu^u, \alpha^u, \gamma^u\})^{\lfloor \lambda/h_m \rfloor + 1} \\ &= \lim_{m \rightarrow +\infty} \left[\left(1 + \frac{3 \max\{\mu^u, \alpha^u, \gamma^u\}}{1/\phi(h_m)} \right)^{1/\phi(h_m)} \right]^{\phi(h_m)(\lfloor \lambda/h_m \rfloor + 1)} \\ &= e^{3 \max\{\mu^u, \alpha^u, \gamma^u\} \lambda}, \end{aligned}$$

there is $C_3 > 0$ such that

$$(53) \quad |C_{n_m, r, \lambda, h_m}| \leq C_3 h_m.$$

Thus

$$\begin{aligned} (54) \quad &B_{n_m, r, \lambda, h_m} - A_{n_m, r, \lambda, h_m} - C_{n_m, r, \lambda, h_m} \\ &\leq |A_{n_m, r, \lambda, h_m}| + |B_{n_m, r, \lambda, h_m}| + |C_{n_m, r, \lambda, h_m}| \\ &\leq (C_1 + C_2 + C_3) h_m, \end{aligned}$$

for all $m \geq M$. Since the right hand side of (54) is independent of n_m, r , we conclude that

$$(55) \quad B_{n_m, r, \lambda, h_m} - A_{n_m, r, \lambda, h_m} - C_{n_m, r, \lambda, h_m} \rightarrow 0,$$

as $m \rightarrow +\infty$, uniformly in r .

On the other hand we note that the C^1 function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ given by

$$(56) \quad f(t) = \beta(t)\varphi(a) + \sigma(t)\psi(b) - \mu(t) - \alpha(t) - \gamma(t)$$

is Riemann-integrable on any bounded interval $I \subset \mathbb{R}_0^+$.

We have that

$$\sum_{k=n_m, r}^{n_m, r + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) + (\lambda - \lfloor \lambda/h_m \rfloor h_m) f(n_m, r h_m + \lfloor \lambda/h_m \rfloor h_m),$$

is a Riemann sum of

$$\int_{n_m, r h_m}^{n_m, r h_m + \lambda} \beta(s)\varphi(a) + \sigma(s)\psi(b) - [\mu(s) + \alpha(s) + \gamma(s)] ds$$

with respect to the partition

$$\{n_m, r h_m, n_m, r h_m + h_m, \dots, n_m, r h_m + \lfloor \lambda/h_m \rfloor h_m, n_m, r h_m + \lambda\}$$

of size h_m of the interval $[n_m, r, n_m, r + \lambda]$. Note that

$$s_{m, r} := (\lambda - \lfloor \lambda/h_m \rfloor h_m) f(n_m, r h_m + \lfloor \lambda/h_m \rfloor h_m) \leq h_m f^u := s_m$$

and $s_m \rightarrow 0$ as $m \rightarrow +\infty$, uniformly in r .

Since f is C^1 with bounded derivative, for any $h > 0$ we have

$$|f(x) - f(x + h)| \leq Ch,$$

where $C = \sup_{t \geq 0} |f'(t)|$. We conclude that

$$(57) \quad \left| \sum_{k=n_{m,r}}^{n_{m,r} + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) + s_{m,r} - \int_{n_{m,r}h_m}^{n_{m,r}h_m + \lambda} f(s) ds \right| \\ < Ch_m^2 \lfloor \lambda/h_m \rfloor + Ch_m^2 \\ < C(\lambda + 1)h_m,$$

thus

$$\sum_{k=n_{m,r}}^{n_{m,r} + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) < \int_{n_{m,r}h_m}^{n_{m,r}h_m + \lambda} f(s) ds - s_{m,r} + C(\lambda + 1)h_m,$$

and therefore

$$(58) \quad \sum_{k=n_{m,r}}^{n_{m,r} + \lfloor \lambda/h_m \rfloor} \phi(h_m) h_m f(kh_m) < \phi(h_m) \left[\int_{n_{m,r}h_m}^{n_{m,r}h_m + \lambda} f(s) ds + C(\lambda + 1)h_m \right].$$

By (58) we conclude that, given $\delta > 0$, there is $r_m \in \mathbb{N}$ such that, for all $r \geq r_m$,

$$(59) \quad \phi(h_m) \sum_{k=n_{m,r_m}}^{n_{m,r_m} + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) < \phi(h_m) [\mathcal{R}_C^u(\lambda) + \delta + C(\lambda + 1)h_m].$$

Finally, recalling that $\mathcal{R}_C^u(\lambda) < 0$, by assumption, by the arbitrariness of $\delta > 0$ and the fact that $h_m \rightarrow 0$ as $m \rightarrow +\infty$, we obtain for sufficiently large $m \in \mathbb{N}$,

$$0 \leq \phi(h_m) \sum_{k=n_{m,r_m}}^{n_{m,r_m} + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) < 0,$$

which is a contradiction. We obtain a).

A similar argument allow us to prove b). In fact, assuming by contradiction that

$$(60) \quad \mathcal{R}_C^\ell(\lambda) > 0,$$

and that there is a sequence $(h_m)_{m \in \mathbb{N}}$ such that $h_m \rightarrow 0$ as $m \rightarrow +\infty$ and

$$\mathcal{R}_D^\ell(\lfloor \lambda/h_m \rfloor, h_m) = \liminf_{n \rightarrow \infty} \prod_{k=n}^{n + \lfloor \lambda/h_m \rfloor} \frac{1 + \beta_k^{h_m} \varphi(a) + \sigma_k^{h_m} \psi(b)}{1 + \mu_k^{h_m} + \alpha_k^{h_m} + \gamma_k^{h_m}} \leq 1,$$

it is possible to conclude that

$$(61) \quad \phi(h_m) \sum_{k=n_{m,r}}^{n_{m,r} + \lfloor \lambda/h_m \rfloor} h_m (\beta(kh_m) \varphi(a) + \sigma(kh_m) \psi(b) - \mu(kh_m) - \alpha(kh_m) - \gamma(kh_m)) \\ < B_{n_{m,r}, \lambda, h_m} - A_{n_{m,r}, \lambda, h_m} + C_{n_{m,r}, \lambda, h_m},$$

where $A_{n_{m,r}, \lambda, h_m}$, $B_{n_{m,r}, \lambda, h_m}$ and $C_{n_{m,r}, \lambda, h_m}$ are given respectively by (46), (47) and (53) and still satisfy (50), (52) and (53). Consequently, given $\delta > 0$, there is $r_m \in \mathbb{N}$ such that, for all $r \geq r_m$,

$$(62) \quad \phi(h_m) \sum_{k=n_{m,r}}^{n_{m,r} + \lfloor \lambda/h_m \rfloor} h_m f(kh_m) > \phi(h_m) \left[\int_{n_{m,r}h_m}^{n_{m,r}h_m + \lambda} f(s) ds - \delta + C(\lambda + 1)h_m \right].$$

Recalling that $\mathcal{R}_C^\ell(\lambda) > 0$, by assumption and since $\delta > 0$ is arbitrary, we obtain for sufficiently large $m \in \mathbb{N}$,

$$0 \geq \phi(h_m) \sum_{k=n_m, r_m}^{n_m, r_m + \lfloor \lambda/h_m \rfloor} f(kh_m) > 0,$$

which is a contradiction. We obtain b) and the theorem follows. \square

Next, for each $L \in \mathbb{N}$, we give an example of a periodic system of period 1 such that the continuous and the discrete time system with time step $h = 1/L$ are not consistent, namely we will have persistence for the continuous time model and extinction for the discrete time model with time step $h = 1/L$.

Example 1. Let $L \in \mathbb{N}$. Consider in system (5) that $\phi(x) = \psi(x) = x$, that, with the exception of σ and β , all parameters are constant, that $\Lambda = \mu$ and that

$$\sigma(t) = \beta(t) = d[1 + c \sin^2(2\pi Lt)(1 + \cos(2\pi t))].$$

We obtain a periodic system of period 1.

In this context, $(x_n, y_n) = (a, b)$, $n \in \mathbb{N}$, and $(x(t), y(t)) = (a, b)$, $t \in \mathbb{R}$, where

$$(a, b) = ((\mu + \eta)/(\mu + \eta + p), p/(\mu + \eta + p)),$$

are respectively solutions of system (9) and system (6). It is now possible to compute the number $\mathcal{R}_C^\ell(1)$. In fact, noting that $x^*(t) + y^*(t) = 1$, we get

$$\begin{aligned} \mathcal{R}_C^\ell(1) &= \int_0^1 \beta(s)x^*(s) + \sigma(s)y^*(s) - \mu - \alpha - \gamma ds \\ &= \int_0^1 d[1 + c \sin^2(2\pi Lt)(1 + \cos(2\pi t))] ds - \mu - \alpha - \gamma \\ &= d(1 + c/2) - \mu - \alpha - \gamma. \end{aligned}$$

We can also compute $\mathcal{R}_D^\ell(1, 1/L)$. Namely we have

$$\mathcal{R}_D^\ell(1, 1/L) = \frac{1 + d/L}{1 + \mu + \alpha + \gamma}.$$

Since

$$\mathcal{R}_C^\ell(1) > 1 \quad \Leftrightarrow \quad \frac{1 + d(1 + c/2)}{1 + \mu + \alpha + \gamma} > 1$$

and

$$\mathcal{R}_D^\ell(1, 1/L) < 1 \quad \Leftrightarrow \quad \frac{1 + d/L}{1 + \mu + \alpha + \gamma} < 1.$$

Letting d be sufficiently small so that $d < \mu + \alpha + \gamma$ (and thus $d < (\mu + \alpha + \gamma)L$) and c be sufficiently large so that $C > 2(\mu + \gamma + \alpha - d)$, we obtain $\mathcal{R}_C^\ell(1) > 1$ and $\mathcal{R}_D^\ell(1, 1/L) < 1$ and we conclude that we don't have consistency for time step $1/L$.

Let $L = 6$ and consider the continuous model with the following parameters $\mu = \Lambda = 0.25$, $\gamma = 0.3$, $\alpha = 0.05$, $\eta = 0.05$, $p = 2/3$, $d = 0.6$ and

$c = 1.5$. In figure 1, we plot function β (or similarly σ) and the component $I(t)$ of the solution of system (5) given by the solver of Mathematica[®] (that we take to represent the solution of the continuous-time model) and the solution of the discrete-time model (8) with time step $1/6$. As can be seen, the infectives are persistent in the continuous-time model but go to extinction in the discrete-time model. we have inconsistency in this case.

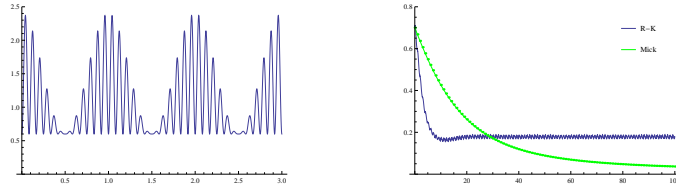


FIGURE 1. Left: function β ; right: inconsistency (time-step= $1/6$).

Note that, changing $\beta(t)$ and $\sigma(t)$ slightly, we can construct an example of a periodic system with period 1 where the infectives in the continuous time model goes to extinction but, in the discrete time model with time step $h = 1/L$, the infectives are persistent.

Furthermore, we emphasise that this lack of consistency is not a result of the discretization method used but simply a result of the fact that the time steps lead to a situation where the points n/L where the functions β and σ are evaluated (in order to obtain the discrete time parameters) correspond to minimums of β and σ .

6. SIMULATION

In this section we do some simulation to illustrate our results. To begin, we compare our model (1) with mass action incidence ($\varphi(S, I) = SI$ and $\psi(V, I) = VI$) with Zhang's model [33]. We use the following set of parameters: $\phi(h) = h + 0.2h^2$, $\Lambda = 0.5$, $\mu(t) = \gamma(t) = \delta(t) = 0.3$, $\alpha(t) = 0.05$, $\eta = 0.05$, $p = 2/3$ and

$$\beta(t) = \sigma(t) = b(1 + 0.3 \cos(t\pi/2)).$$

Setting $b = 0.3$ we obtain $\mathcal{R}_C^u(4) = -0.6 < 0$ and thus we conclude that we have extinction for the continuous model. Taking time-steps equal to 4, 1 and 0.5, we get $\mathcal{R}_D^u(0, 4) = \mathcal{R}_D^\ell(0, 4) = 1$, $\mathcal{R}_D^u(3, 1) = 0.644 < 1$ and $\mathcal{R}_D(7, 0.5) = 0.601 < 1$ and we conclude that we have extinction for time steps 1 and 0.5. For these parameters, we have consistency in the sense of Theorem 5 as long as the time step is less than 0.05. Clearly, there is numerical evidence that there is consistency even for higher time steps. Figure 2 illustrates this situation.

Changing β to 0.9 we obtain $\mathcal{R}_C^\ell(4) = 3.4 > 0$ and thus we conclude that we have persistence. Taking time-steps equal to 2, 1 and 0.5, we get

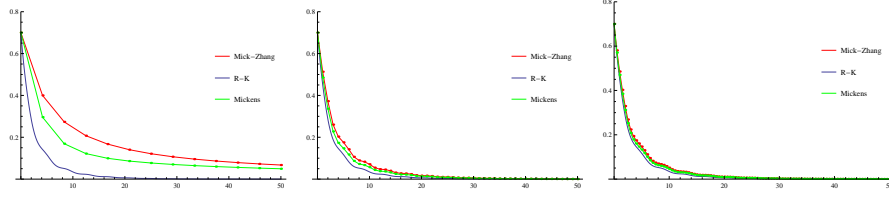


FIGURE 2. SI ; $\phi(h) = h + 0.2h^2$; step-size: 4, 1, 0.5.

$\mathcal{R}_D^\ell(1, 2) = 3.201 > 1$, $\mathcal{R}_D^\ell(3, 1) = 5.9 > 1$ and $\mathcal{R}_D^\ell(7, 0.5) = 10.2 > 1$ and we conclude that we have persistence for all these time steps. Figure 3 illustrates this situation. Figure 2 and figure 3 suggest that numerically our model is slightly better than Zhang's model, at least for large time steps.

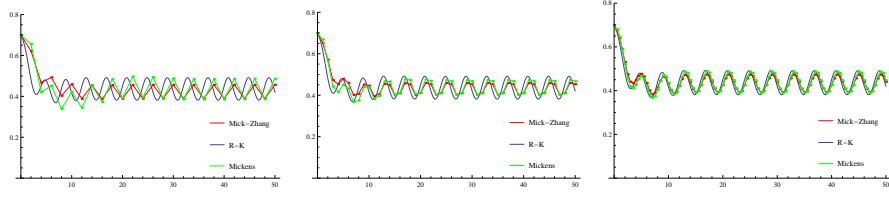


FIGURE 3. SI ; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

Next, we compare our model with the discretized model obtained by Euler method and the output of the Mathematica[®] solver ODE (that uses a Runge-Kutta method). Considering $\beta = 0.3$, we get extinction for the continuous time model, as we already saw. Taking time-steps equal to 2, 1 and 0.5, we can see in figure 4 that for all methods considered and all time steps we have extinction, although the behaviour of our model shadows better the behaviour given by Mathematica's solver, at least for these time steps.

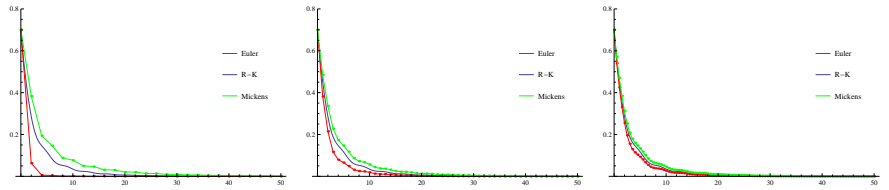
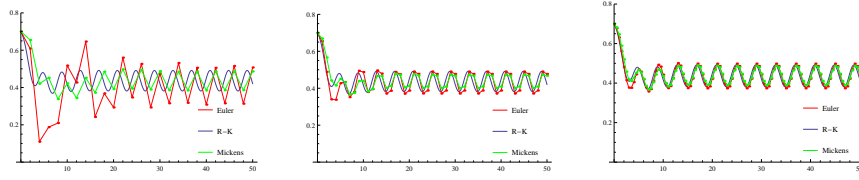


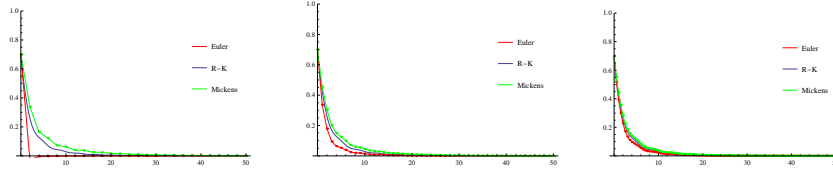
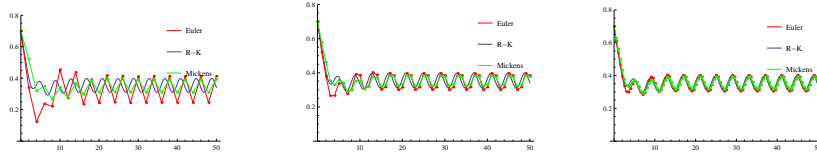
FIGURE 4. SI ; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

Changing β to 0.9 we already saw that we obtain persistence for the continuous model. Figure 5 illustrates this situation

Next, we change our incidence function and consider $\phi(S, I) = SI/(1 + 0.7I)$, maintaining the set of parameters. Letting $\beta = 0.3$ we obtain we have extinction for the continuous model and letting $\beta = 0.9$ we obtain we

FIGURE 5. SI ; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

have persistence for the continuous time model. Note that the thresholds \mathcal{R}_C^u , \mathcal{R}_C^ℓ , \mathcal{R}_D^u and \mathcal{R}_D^ℓ are similar to the mass action case. Figures 6 and 7 illustrate this situation.

FIGURE 6. SI ; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.FIGURE 7. SI ; $\phi(h) = h + 0.2h^2$; step-size: 2, 1, 0.5.

Doing corresponding simulations and comparisons for our model with $\phi(h) = (1 - e^{-0.002h})/(0.002)$ instead of $\phi(h) = h + 0.2h^2$ we can draw the same conclusions regarding extinction/persistence and relation to Zhang's model and the model obtained by Euler method.

7. CONCLUSIONS

In this paper we considered a discretization procedure, based on Mickens discretization Method, to obtain a discrete-time model from a continuous time with vaccination and incidence given by a general function. For a family of models containing the obtained discrete-time model, we obtain results on the persistence and the extinction of the disease (Theorems 3 and 4) that contain the results of Zhang [33] as a particular case. Our

threshold conditions depend on the parameters of the model and also on the derivative of the incidence function, with respect to the infectives, computed on some disease-free solution. This agrees with the continuous counterparts of these results [22].

We also considered the problem of establishing the consistency of the continuous-time model and the discrete-time model for small time-steps, in the sense that if the time step is small enough when we have persistence (respectively extinction) for the continuous-time model we also have persistence (respectively extinction) for the discrete-time model (at least for situations where Theorems 1 and 2 allow us to conclude that we have persistence or extinction). Assuming that the parameters are differentiable, our result on this direction, Theorem 5, furnishes an interval $[0, a]$, with a depending only on the parameters of the model and their derivatives, where there is consistency. We also give an example of a periodic system of period 1 such that the continuous and the discrete time system with time step $h = 1/L$ are not consistent, namely we will have persistence for the continuous time model and extinction for the discrete time model with time step $h = 1/L$. These examples show the importance of knowing that for time steps smaller than some explicit value we have consistency, a type of result like the one in Theorem 5.

Finally, we carried out some simulations to illustrate our results. As one might expect our simulations furnish evidence that we may have consistency in intervals with length several times bigger than the length of interval given in Theorem 5.

REFERENCES

- [1] L. J. S. Allen, M. A. Jones, C. F. Martin, A discrete-time model with vaccination for a measles epidemic, *Math. Biosci.*, 105 (1991), 111-131
- [2] L. J. S. Allen, Some discrete-time si,sir and sis epidemic models, *Math. Biosci.*, 124 (1994), 83-105
- [3] K. L. Cooke, A discrete-time epidemic model with classes of infectives and susceptibles, *Theor. Popul. Biol.*, 7(1975), 175-196
- [4] Qianqian Cui and Qiang Zhang, Global stability of a discrete SIR epidemic model with vaccination and treatment, *Journal of Difference Equations and Applications*, 2 (2015), 111-117
- [5] M. C. M. DeJong, O. Diekmann and J. A. P. Heesterbeek, The computation of r_0 for discrete-time epidemic models with dynamic heterogeneity, *Math. Biosci.*, 119 (1994), 97-114
- [6] O. Diekmann and J. A. P. Heesterbeek, (2000), *Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation*, Wiley, Chichester
- [7] S. N. Elaydi, *An Introduction to Difference Equations*, third ed., Springer, NewYork, 2005
- [8] Khalid Hattaf, Abid Ali Lashari, Brahim El Boukari and Noura Yousfi, Effect of Discretization on Dynamical Behavior in an Epidemiological Model, *Differ Equ Dyn Syst*, 23 (2015), 403-413

- [9] Hua, Teng and Zhang, Threshold conditions for a discrete nonautonomous SIRS model, *Mathematics and Computers in Simulation*, 97 (2014), 80-93
- [10] Hu, Teng, Jia, Zhang and Zhang, Dynamical analysis and chaos control of a discrete SIS epidemic model, *Advances in Difference Equations* 2014, 2014:58
- [11] V. Kaitala, M. Heino and W. M. Getz, Host-parasite dynamics and the evolution of host immunity and parasite fecundity strategies, *Bull. Math. Biol.*, 59 (1997), 427-450
- [12] Atul A. Khasnis and Mary D. Nettleman, Global Warming and Infectious Disease, *Archives of Medical Research*, 36 (2005), 689-696
- [13] Peter E. Kloeden and Christia Pötzsche, Nonautonomous dynamical systems in the life sciences, *Lecture Notes in Mathematics* 2102 (2013)
- [14] G. Lena and G. Serio, A discrete method for the identification of parameters of a deterministic epidemic model, *Math. Biosci.*, 60 (1982), 161-175
- [15] C. Lefevre and M. Malice, A discrete time model for a s-i-s infectious disease with a random number of contacts between individuals, *Math.Model.*, 7 (1986), 785-792
- [16] C. Lefevre and P. Picard, On the formulation of discrete-time epidemic models, *Math.Biosci.*, 95 (1989), 27-35
- [17] Liu, Peng and Zhang, Effect of discretization on dynamical behavior of SEIR and SIR models with nonlinear incidence, *Applied Mathematics Letters*, 39 (2015), 60-66
- [18] I. M. Longini, The generalized discrete-time epidemic model with immunity: a synthesis, *Math. Biosci.*, 82 (1986), 19-41
- [19] Mickens, Nonstandard finite difference schemes for differential equations, *J. Difference Equ. Appl.* 8(2002), 823-847
- [20] R. E. Mickens, A discrete-time model for the spread of periodic diseases without immunity, *Biosystems*, 26 (1992), 193-198
- [21] R. E. Mickens, Discretizations of nonlinear differential equations using explicit nonstandard methods, *J.Comput.Appl.Math.*, 110 (1999), 181-185
- [22] E. Pereira, C. M. Silva and J. A. L. Silva, A Generalized Non-Autonomous SIRVS Model, *Math. Methods Appl. Sci.*, 36 (2013), 275-289
- [23] M. Sekiguchi, Permanence of a discrete sirs epidemic model with time delays, *Appl. Math. Lett.* 23 (2010), 1280-1285
- [24] M. Sekiguchi and E. Ishiwata, Global dynamics of a discretized sir sepidemic model with time delay, *J. Math. Anal. Appl.*, 371 (2010), 195-202
- [25] C. C. Spicer, The mathematical modeling of influenza, *Brit. Med. Bull.*, 35 (1979), 23-28
- [26] Tan, Gao and Fan, Bifurcation Analysis and Chaos Control in a Discrete Epidemic System, *Discrete Dynamics in Nature and Society*, Volume 2015, Article ID 974868
- [27] D. P. L. Viaud, Large deviations for discrete-time epidemic models, *Math. Biosci.*, 117 (1993), 197-210
- [28] J. Xu, Z. Teng and S. Gao, Almost sufficient and necessary conditions for permanence and extinction of nonautonomous discrete logistic systems with time-varying delays and feedback control, *Appl. Math-Czech.*, 56 (2011), 207-225

- [29] , Wang, Teng and Rehim, Lyapunov Functions for a Class of Discrete SIRS Epidemic Models with Nonlinear Incidence Rate and Varying Population Sizes, Discrete Dynamics in Nature and Society, Volume 2014, Article ID 472746
- [30] Wei and Le, Existence and Convergence of the Positive Solutions of a Discrete Epidemic Model, Discrete Dynamics in Nature and Society, Volume 2015, Article ID 434537
- [31] R. W. West and J. R. Thompson, Models for the simple epidemic, Math. Biosci., 141 (1997), 29-39
- [32] T. Zhang, Z. Teng and S. Gao, (2008), Threshold conditions for a nonautonomous epidemic model with vaccination, Applicable Analysis, 87, 181-199.
- [33] T. Zhang, Permanence and extinction in a nonautonomous discrete SIRVS epidemic model with vaccination, Appl. Math. Comput., 271 (2015), 716-729
- [34] T. Zhang, Z. Teng and S. Gao, Threshold conditions for a non-autonomous epidemic model with vaccination, Appl. Anal., 87 (2008), 181-199
- [35] T. Zhang, Z. Teng, An SIRVS epidemic model with pulse vaccination strategy, J. Theoret. Biol., 250 (2008), 375-381
- [36] T. Zhang, J. Liu and Z. Teng, A non-autonomous epidemic model with time delay and vaccination, Math. Methods Appl. Sci., 33 (2010), 1-11
- [37] T. Zhang, J. Liu and Z. Teng, Threshold conditions for a discrete nonautonomous SIR Smodel, Math. Meth. Appl. Sci., 38 (2015), 1781-1794
- [38] Zhou, Li and Wang, Bifurcations for a deterministic SIR epidemic model in discrete time, Advances in Difference Equations 2014, 2014:168

J. MATEUS, INSTITUTO POLITÉCNICO DA GUARDA, 6300-559 GUARDA, PORTUGAL

E-mail address: `jmateus@ipg.pt`

C. SILVA, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DA BEIRA INTERIOR, 6201-001 COVILHÃ, PORTUGAL

E-mail address: `csilva@ubi.pt`

URL: `www.mat.ubi.pt/~csilva`

S. VAZ, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DA BEIRA INTERIOR, 6201-001 COVILHÃ, PORTUGAL

E-mail address: `svaz@ubi.pt`